Cauchy's theorem (group)

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Reminder

Let G be a group, X be a set and $G \circlearrowright X$.

Definition

For $x \in X$,

- $\mathrm{Orb}(x) := \{g \cdot x \mid g \in G\}$
- $\mathrm{Stab}(x) := \{g \in G \mid g \cdot x = x\}$

Lagrange's theorem

If $|G| = n$ and $H \leqslant G$ with $|H| = d$, then $d \mid n$.

Orbit stabilizer's formula

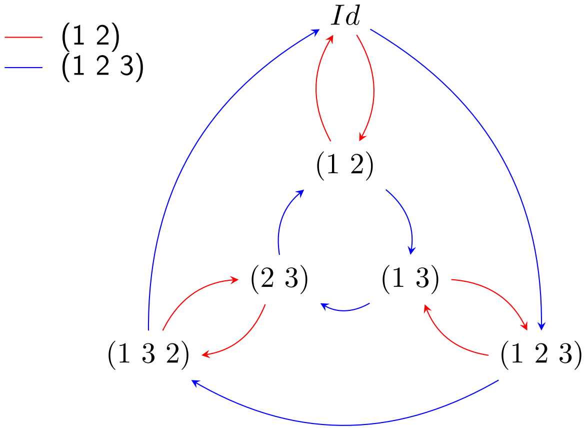
If G is finite, for all $x \in X$, we have

$$|G| = |\mathrm{Stab}(x)| \times |\mathrm{Orb}(x)|.$$

Theorem

Let G be a finite group and p be a prime. If p divides the order of G, then G has an element of order p.

Cauchy's theorem / Example



[1]

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[1] Cayley's graph of $\mathfrak{S}_3$.

Let $(G, \star, \mathbf{e})$ be a finite group and p be a prime such that p divides $|G|$.
Let's take $X = \{(g_1, \ldots, g_p) \in G^p \mid g_1 \star \cdots \star g_p = \mathbf{e}\}$.
Note that $|X| = |G|^{p-1}$.
We will study the cyclic action of $\mathbb{Z}/p\mathbb{Z}$ on X given by

$$(g_1, g_2, \ldots, g_p) \longmapsto (g_p, g_1, \ldots, g_{p-1}).$$

Obviously, we have :

$$X = \biguplus_i \mathsf{Orb}(x_i),$$

then :

$$|X| = \sum_i |\mathsf{Orb}(x_i)|.$$

We obtain :

$$\sum_i |\mathsf{Orb}(x_i)| = 0 \pmod{p}. \tag{1}$$

Now, let's look at the stabilizers of $x \in X$. By Lagrange's theorem, we have two types of stabilizers : $\mathbb{Z}/p\mathbb{Z}$ or $\{0\}$. Therefore, for the first type of stabilizer, the orbits are of the form $\{(g, \ldots, g)\}$. In the second case, p divide $|\mathrm{Orb}(x)|$ (with orbit stabilizer's formula), so we can rewrite (1) without them :

$$\sum_j |\mathrm{Orb}(x_j)| = 0 \pmod{p}.$$

It is a non-zero sum, because it has $(\mathbf{e}, \ldots, \mathbf{e})$.
Thus, we have at least $p - 1$ elements of order p in G.

$\square$[2]

[2] *For an another proof : McKay 1959; V. and Tagne 2021.*

Conclusion

Theorem

Let G be a finite group and p be a prime. If p divides the order of G, then G has an element of order p.

Opening

What are the possible extensions or generalizations of Cauchy's theorem that we could explore ?

References

McKay, James H. (Feb. 1959). "ANOTHER PROOF OF CAUCHY'S GROUP THEOREM". In: *American Mathematical Monthly* 66. Seattle University, p. 119. URL: http://www.cs.toronto.edu/~yuvalf/McKay%20Another%20Proof%20of%20Cauchy's%20Group%20Theorem.pdf.

V., Christian and Nguembou Tagne (Aug. 2021). *Cauchy's theorem for groups*. URL: https://formalismathematica.files.wordpress.com/2021/08/cauchy_groupes.pdf.